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Invariants of Unitary Reflection Groups

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有限ユニタリ鏡映群は、Shephard-Todd [3] によって分類されていて、1〜37までの名前が与えられている。本稿では、それらの不変量の表を作り、そこから得られるobservationを紹介したい。

§1. Notations and Table

1. The names of finite unitary reflection groups are due to Shephard-Todd [3],
2. $m_1, \cdots, m_\ell$ are the exponents of group (i.e., the degrees of basis of invariant polynomials -1); $(m_1 \le m_2 \le \cdots \le m_\ell)$
   $n_1, \cdots, n_\ell$ are the exponents of arrangement (i.e., the degrees of basis of logarithmic derivations $+1$). (See [4]) $(n_1 \le n_2 \le \cdots \le n_\ell)$
3. $m = \sum_{i=1}^{\ell} m_i$, $n = \sum_{i=1}^{\ell} n_i$.

4. $r_i = \#\{\text{reflections of order } i\}$,

   $s_i = \#\{$reflecting hyperplane H such that there exist exactly i fixing H pointwise$\}$

5. $(d_1, \cdots, d_\ell)$ is <u>self-dual</u> $\underset{\text{def}}{\Longleftrightarrow}$ $d_i + d_{\ell+1-i} = \text{const.}$ $(1 \leq i \leq \ell)$

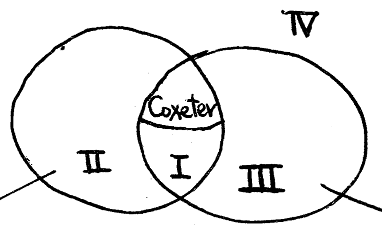
6. $(m_1, \cdots, m_\ell)$ and $(n_1, \cdots, n_\ell)$ are <u>codual</u>

$$\Updownarrow \text{def}$$

$$m_i + n_{\ell+1-i} = \text{const.} \quad (1 \leq i \leq \ell)$$

7. $t := \dfrac{2n}{\ell}$, $\quad H := \dfrac{2(m+n)}{2\ell + (m-n)}$

8. type:



generated by reflections of order 2

generated by ℓ reflections

9. Condition (A): Are (m_i) & (n_i) both self-dual?

   Condition (B): If yes, $m_1 + m_\ell = n_1 + n_\ell$?

   Condition (C): Are (m_i) & (n_i) codual?

   Condition (D): Is G generated by ℓ reflections?

   Condition (E): Is G generated by reflections of order 2?

Group	$m_1, \cdots, m_\ell$ $n_1, \cdots, n_\ell$	m n	$r_2, r_3, \cdots$ $S_2, S_3, \cdots$	t H	$m_\ell+1$ $n_\ell+1$	(A)	(B)	(C)	(D)	(E)	type
1 (A_ℓ)	$1, 2, \cdots, \ell$ $1, 2, \cdots, \ell$	$\frac{\ell}{2}(\ell+1)$ $\frac{\ell}{2}(\ell+1)$	$\frac{\ell}{2}(\ell+1), 0, 0, \cdots$ $\frac{\ell}{2}(\ell+1), 0, 0, \cdots$	$\ell+1$ $\ell+1$	$\ell+1$ $\ell+1$	y	y	y	y	y	Coxeter
2 $G(r,p,\ell)$ $r=pq$											
$r=q=2$ (B_ℓ)	$1, 3, 5, \cdots, 2\ell-1$ $1, 3, 5, \cdots, 2\ell-1$	ℓ^2 ℓ^2	$\ell^2, 0, 0, \cdots$ $\ell^2, 0, 0, \cdots$	2ℓ 2ℓ	2ℓ 2ℓ	y	y	y	y	y	Coxeter
$r=4$ $p=q=2$	$3, 7, \cdots, 4\ell-5 (=m_\ell), 2\ell-1$ $1, 5, \cdots, 4\ell-3$	$\ell(2\ell-1)$ $\ell(2\ell-1)$	$\ell(2\ell-1), 0, 0, \cdots$ $\ell(2\ell-1), 0, 0, \cdots$	$4\ell-2$ $4\ell-2$	$4\ell-4$ $4\ell-2$	y	y	n	n	y	II
$r=q>2$	$r-1, 2r-1, \cdots, \ell r-1$ $1, r+1, \cdots, (\ell-1)r+1$	$\ell\left(\frac{(\ell+1)r}{2}-1\right)$ $\ell\left(\frac{(\ell-1)r}{2}+1\right)$		$(\ell-1)r+2$ 2ℓ	ℓr $(\ell-1)r+2$	y	n	y	y	n	III
$r=2q>4$	$r-1, 2r-1, \cdots, (\ell-1)r-1, \frac{\ell r}{2}-1$ $1, r+1, \cdots, (\ell-1)r+1$	$\ell\left(\frac{\ell r}{2}-1\right)$ $\ell\left(\frac{(\ell-1)r}{2}+1\right)$		$(\ell-1)r+2$ $4\ell-2$	$(\ell-1)r$ $(\ell-1)r+2$	y	n	n	n	n	IV
$q\neq r\neq 2q$ $q>2$	$r-1, 2r-1, \cdots, (\ell-1)r-1, \ell q-1$ $1, r+1, \cdots, (\ell-1)r+1$	$\ell\left(\frac{(\ell-1)r}{2}+q-1\right)$ $\ell\left(\frac{(\ell-1)r}{2}+1\right)$		$(\ell-1)r+2$ $2(\ell-1)p+2$	$(\ell-1)r$ $(\ell-1)r+2$	n	-	n	n	n	IV
$q\neq r\neq 2q$ $q=2$	$r-1, 2r-1, \cdots, (\ell-1)r-1, 2\ell-1$ $1, r+1, \cdots, (\ell-1)r+1$	$\ell\left(\frac{(\ell-1)r}{2}+1\right)$ $\ell\left(\frac{(\ell-1)r}{2}+1\right)$		$(\ell-1)r+2$ $(\ell-1)r+2$	$(\ell-1)r$ $(\ell-1)r+2$	n	-	n	n	y	II
$p=r>2$ $\ell>2$	$r-1, 2r-1, \cdots, (\ell-1)r-1, \ell-1$ $1, r+1, \cdots, (\ell-2)r+1, (\ell-1)(r-1)$	$\frac{\ell(\ell-1)r}{2}$ $\frac{\ell(\ell-1)r}{2}$	$\frac{\ell(\ell-1)r}{2}, 0, 0, \cdots$ $\frac{\ell(\ell-1)r}{2}, 0, 0, \cdots$	$(\ell-1)r$ $(\ell-1)r$	$(\ell-1)r$ $(\ell-1)(r-1)+1$	n	-	y	y	y	I
$p=r=2$ (D_ℓ)	$1, 3, 5, \cdots, 2\ell-3, \ell-1$ $1, 3, 5, \cdots, 2\ell-3, \ell-1$	$\ell(\ell-1)$ $\ell(\ell-1)$	$\ell(\ell-1), 0, 0, \cdots$ $\ell(\ell-1), 0, 0, \cdots$	$2(\ell-1)$ $2(\ell-1)$	$2(\ell-1)$ $2(\ell-1)$	y	y	y	y	y	Coxeter

Group	$m_1, \cdots, m_\ell$ $n_1, \cdots, n_\ell$	m n	$r_2, r_3, \cdots$ $s_2, s_3, \cdots$	t H	$m_\ell+1$ $n_\ell+1$	(A)	(B)	(C)	(D)	(E)	type
3 $r>2$	$r-1$ 1	$r-1$ 1	$0,0,\cdots,0,1,0,\cdots$	2 2	r 2	y	n	y	y	n	III
4	3,5 1,3	8 4	0,8,0, ⋯ 0,4,0, ⋯	4 3	6 4	y	n	y	y	n	III
5	5,11 1,7	16 8	0,16,0, ⋯ 0,8,0, ⋯	8 4	12 8	y	n	y	y	n	III
6	3,11 1,9	14 10	6,8,0, ⋯ 6,4,0, ⋯	10 6	12 10	y	n	y	y	n	III
7	11,11 1,13	22 14	6,16,0, ⋯ 6,8,0, ⋯	14 6	12 14	y	n	n	n	n	IV
8	7,11 1,5	18 6	6,0,12, ⋯ 0,0,6, ⋯	6 3	12 6	y	n	y	y	n	III
9	7,23 1,17	30 18	18,0,12, ⋯ 12,0,6, ⋯	18 6	24 18	y	n	y	y	n	III
10	11,23 1,13	34 14	6,16,12, ⋯ 0,8,6, ⋯	14 4	24 14	y	n	y	y	n	III

Group	$m_1, \cdots, m_e$ $n_1, \cdots, n_e$	m n	$r_2, r_3, \cdots$ $s_2, s_3, \cdots$	t H	m_e+1 n_e+1	(A)	(B)	(C)	(D)	(E)	type
11	23,23 1,25	46 26	18,16,12,… 12,8,6,…	26 6	24 26	y	n	n	n	n	IV
12	5,7 1,11	12 12	12,0,… 12,0,…	12 12	8 12	y	y	n	n	y	II
13	7,11 1,17	18 18	18,0,… 18,0,…	18 18	12 18	y	y	n	n	y	II
14	5,23 1,19	28 20	12,16,… 12,8,…	20 8	24 20	y	n	y	y	n	III
15	11,23 1,25	34 26	18,16,… 18,8,…	26 10	24 26	y	n	n	n	n	IV
16	19,29 1,11	48 12	0,0,0,48 0,0,0,12	12 3	30 12	y	n	y	y	n	III
17	19,59 1,41	78 42	30,0,0,48 30,0,0,12	42 6	60 42	y	n	y	y	n	III
18	29,59 1,31	88 32	0,40,0,48 0,20,0,12	32 4	60 32	y	n	y	y	n	III
19	59,59 1,61	118 62	30,40,0,48 30,20,0,12	62 6	60 62	y	n	n	n	n	IV
20	11,29 1,19	40 20	0,40,… 0,20,…	20 5	30 20	y	n	y	y	n	III

Group	$m_1, \cdots, m_\ell$ $n_1, \cdots, n_\ell$	m n	$r_2, r_3, \cdots$ $s_2, s_3, \cdots$	t H	$m_\ell+1$ $n_\ell+1$	(A)	(B)	(C)	(D)	(E)	type
21	11, 59 1, 49	70 50	30, 40, ⋯ 30, 20, ⋯	50 10	60 50	y	n	y	y	n	III
22	11, 19 1, 29	30 30	30, 0, ⋯ 30, 0, ⋯	30 30	20 30	y	y	n	n	y	II
23 (H_3)	1, 5, 9 1, 5, 9	15 15	15, 0, ⋯ 15, 0, ⋯	10 10	10 10	y	y	y	y	y	I
24	3, 5, 13 1, 9, 11	21 21	21, 0, ⋯ 21, 0, ⋯	14 14	14 12	n	-	y	y	y	I
25	5, 8, 11 1, 4, 7	24 12	0, 24, ⋯ 0, 12, ⋯	8 4	12 8	y	n	y	y	n	III
26	5, 11, 17 1, 7, 13	33 21	9, 24, ⋯ 9, 12, ⋯	14 6	18 14	y	n	y	y	n	III
27	5, 11, 29 1, 19, 25	45 45	45, 0, ⋯ 45, 0, ⋯	30 30	30 26	n	-	y	y	y	I.
28 (F_4)	1, 5, 7, 11 1, 5, 7, 11	24 24	24, 0, ⋯ 24, 0, ⋯	12 12	12 12	y	y	y	y	y	Coxeter
29	3, 7, 11, 19 1, 9, 13, 17	40 40	40, 0, ⋯ 40, 0, ⋯	20 20	20 18	n	-	y	y	y	I
30 (H_4)	1, 11, 19, 29 1, 11, 19, 29	60 60	60, 0, ⋯ 60, 0, ⋯	30 30	30 30	y	y	y	y	y	Coxeter

Groups	$m_1, \cdots, m_\ell$ $n_1, \cdots, n_\ell$	m n	$r_2, r_3, \cdots$ $s_2, s_3, \cdots$	t H	$m_\ell + 1$ $n_\ell + 1$	(A)	(B)	(C)	(D)	(E)	type
31	7, 11, 19, 23 1, 13, 17, 29	60 60	60, 0, ⋯ 60, 0, ⋯	30 30	24 30	Y	Y	n	n	Y	II
32	11, 17, 23, 29 1, 7, 13, 19	80 40	0, 80, ⋯ 0, 40, ⋯	20 5	30 20	Y	n	Y	Y	n	III
33	3, 5, 9, 11, 17 1, 7, 9, 13, 15	45 45	45, 0, ⋯ 45, 0, ⋯	18 18	18 16	n	-	Y	Y	Y	I
34	5, 11, 17, 23, 29, 41 1, 13, 19, 25, 31, 37	126 126	126, 0, ⋯ 126, 0, ⋯	42 42	42 38	n	-	Y	Y	Y	I
35 (E_6)	1, 4, 5, 7, 8, 11 1, 4, 5, 7, 8, 11	36 36	36, 0, ⋯ 36, 0, ⋯	12 12	12 12	Y	Y	Y	Y	Y	Coxeter
36 (E_7)	1, 5, 7, 9, 11, 13, 17 1, 5, 7, 9, 11, 13, 17	63 63	63, 0, ⋯ 63, 0, ⋯	18 18	18 18	Y	Y	Y	Y	Y	Coxeter
37 (E_8)	1, 7, 11, 13, 17, 19, 23, 29 1, 7, 11, 13, 17, 19, 23, 29	120 120	120, 0, ⋯ 120, 0, ⋯	30 30	30 30	Y	Y	Y	Y	Y	Coxeter

§ 2. Observations

1. (Orlik-Solomon [1])

(m_i) & (n_i) are codual (condition (C))

$$\Updownarrow$$

G is generated by l reflections (condition (D))

$$\Updownarrow$$

$$m_l \geq n_l$$

2. H is always a natural number. (This invariant was introduced by T. Yano.)
t has been proved to be a natural number [5].

3. There are five cases:

	(A)	(B)	(C)	(D)	(E)		
$t = H = m_l + 1 = n_l + 1 \Longleftrightarrow$	y	y	y	y	y	$\Longleftrightarrow$	Coxeter
$t = H = m_l + 1 > n_l + 1 \Longleftrightarrow$	n	-	y	y	y	$\Longleftrightarrow$	I
$t = H = n_l + 1 > m_l + 1 \Longleftrightarrow$	y	y	n	n	y	$\Longleftrightarrow$	II
$m_l + 1 > n_l + 1 = t > H \Longleftrightarrow$	y	n	y	y	n	$\Longleftrightarrow$	III
$n_l + 1 = t > m_l + 1 > H \Longleftrightarrow$	y	n	n	n	n	$\Longleftrightarrow$	IV

(with only exception: 2)

4. III $\Longleftrightarrow$ Shephard group (Orlik-Solomon [2])

References

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